The Nuggetizer: Abstracting Away Higher-Orderness for Program Verification

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Objective

Prove non-trivial *inductive* properties about *higher-order* programs

- Statically
- Automatically
- Without any programmer annotations

Exemplar: Value range analysis for higher-order functional programs

- Inferring the range of values assignable to integer variables at runtime

Example: Factorial Program

```
let f = λfact. λn. if (n != 0) then
                        n * fact fact (n - 1)
                      else  1
in f f 5
```

Recursion encoded by "self-passing"

Focus of rest of the talk: Verify range of n is [0, 5]

Motivation

Higher-Order Functional Programming

- Powerful programming paradigm
- Complex from automated verification standpoint
  - Actual low-level operations and the order in which they take place are far removed from the source code, especially in presence of recursion, for example, via the Y-combinator

The simpler first-order view is easiest for automated verification methods to be applied to

Our Approach

- Abstract Away the Higher-Orderness
  - Distill the first-order computational structure from higher-order programs into a *nugget*
  - Preserve much of other behavior, including
    - Control-Flow (Flow-Sensitivity + Path-Sensitivity)
    - Infinite Datatype Domains
    - Other Inductive Program Structures
- Feed the nugget to a theorem prover to prove desirable properties of the source program

A Nugget

- Set of purely first-order inductive definitions
- Denotes the underlying computational structure of the higher-order program
  - Characterizes all value bindings that may arise during corresponding program's execution
- Extracted automatically by the *nuggetizer* from any untyped functional program

Example: Factorial Program

```
let f = λfact. λn. if (n != 0) then
                          n * fact fact (n - 1)
                        else  1
in f f 5
```

Property of interest: Range of n is [0, 5]

Nugget at n: $\{ n \mapsto 5, n \mapsto (n - 1)^{n\, != \,0} \}$

Example: Factorial Program

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let f = λfact. λn. if (n != 0) then
                        n * fact fact (n - 1)
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in f f 5
```

Property of interest: Range of n is [0, 5]

Nugget at n: $\{\ n \mapsto 5,\ n \mapsto (n - 1)^{n\ !=\ 0}\ \}$

Example: Factorial Program

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let f = λfact. λn. if (n != 0) then
                        n * fact fact (n - 1)
                      else  1
in f f 5
```

Property of interest: Range of n is [0, 5]

Nugget at n: $\{\ n \mapsto 5,\ n \mapsto (n - 1)^{n\ !=\ 0}\ \}$

Guard: A precondition on the usage of the mapping

Denotation of a Nugget

The least set of values implied by the mappings such that their guards hold

$$\{ n \mapsto 5, n \mapsto (n - 1)^{n != 0} \}$$

$$\Downarrow$$

$$\{ n \mapsto 5, n \mapsto 4, n \mapsto 3, n \mapsto 2, n \mapsto 1, n \mapsto 0 \}$$

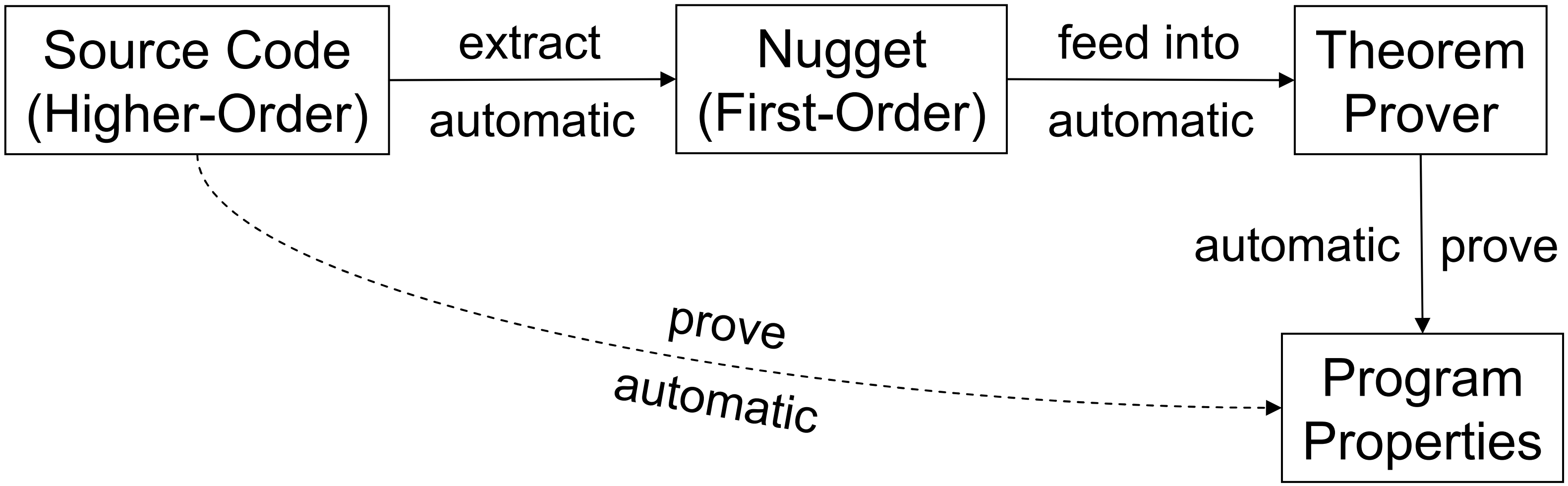
$n \mapsto -1$ is disallowed as $n \mapsto 0$ does not satisfy the guard (n != 0), analogous to the program's computation

Range of n is denoted to be *precisely* [0, 5]

Nuggets in Theorem Provers

- Nuggets are automatically translatable to equivalent definitions in a theorem prover
  - Theorem provers provide built-in mechanisms for writing inductive definitions, and automatically generating proof strategies thereupon
- We provide an automatic translation scheme for Isabelle/HOL
  - We have proved $0 \leq n \leq 5$ and similar properties for other programs

Summary of Our Approach



The Nuggetizer

- Extracts nuggets from higher-order programs via a collecting semantics
  - Incrementally accumulates the nugget over an abstract execution of the program
- = 0CFA + flow-sensitivity + path-sensitivity
  - Abstract execution closely mimics concrete execution
  - Novel *prune-rerun* technique ensures convergence and soundness in presence of flow-sensitivity and recursion

Illustration of the Nuggetizer

```
let f = λfact. λn. let r = if (n != 0) then
                               let fact′ = fact fact in
                               let r′ = fact′ (n - 1) in
                                  n * r′
                             else  1
                        in r
in let f′ = f f in
in let z = f′ 5 in
   z
```

Abstract Call Stack

empty

Abstract Environment

A-normal form – each program point has an associated variable

Illustration of the Nuggetizer

```
let f = λfact. λn. let r = if (n != 0) then
                              let fact′ = fact fact in
                              let r′ = fact′ (n - 1) in
                                n * r′
                            else  1
                       in r
in let f′ = f f in
in let z = f′ 5 in
   z
```

redex

Abstract Call Stack

empty

Abstract Environment

$f \mapsto (\lambda fact.\ \lambda n.\ \ldots)$

Collect the let-binding in the abstract environment

Illustration of the Nuggetizer

```
let f = λfact. λn. let r = if (n != 0) then
                                 let fact′ = fact fact in
                                 let r′ = fact′ (n - 1) in
                                    n * r′
                               else  1
                         in r
in let f′ = f f in
in let z = f′ 5 in
   z
```

redex

Abstract Call Stack

(λfact. λn. …)

Abstract Environment

f ↦ (λfact. λn. …), fact ↦ f

Invoke (λfact. λn. …) on f, and place it in the call stack

Illustration of the Nuggetizer

let f = λfact. λn. let r = if (n != 0) then
let fact′ = fact fact in
let r′ = fact′ (n - 1) in
n * r′
else 1
in r
in let f′ = f f in
in let z = f′ 5 in
z

redex

Abstract Call Stack

empty

Abstract Environment

f ↦ (λfact. λn. …), fact ↦ f, f′ ↦ (λn. …)

Pop (λfact. λn. …), and return (λn. …) to f′

Illustration of the Nuggetizer

```
let f = λfact. λn. let r = if (n != 0) then
                                let fact′ = fact fact in
                                let r′ = fact′ (n - 1) in
                                   n * r′
                              else  1
                        in r
in let f′ = f f in
in let z = f′ 5 in
   z
```

redex

Abstract Call Stack

(λn. …)

Abstract Environment

f ↦ (λfact. λn. …), fact ↦ f, f′ ↦ (λn. …),

n ↦ 5

Invoke (λn. …) on 5, and place it in the call stack

Illustration of the Nuggetizer

let f = λfact. λn. let r = if (n != 0) then
let fact′ = fact fact in
let r′ = fact′ (n - 1) in
n * r′
else 1
in r
in let f′ = f f in
in let z = f′ 5 in
z

redex

Abstract Call Stack

(λn. …)

Abstract Environment

f ↦ (λfact. λn. …), fact ↦ f, f′ ↦ (λn. …),

n ↦ 5

Analyze the *then* and *else* branches in parallel

Illustration of the Nuggetizer

redex

```
let f = λfact. λn. let r = if (n != 0) then
                               let fact′ = fact fact in
                               let r′ = fact′ (n - 1) in
                                 n * r′
                             else  1
                       in r
in let f′ = f f in
in let z = f′ 5 in
   z
```

Abstract Call Stack

(λn. …)

(λfact. λn. …)

Abstract Environment

f $\mapsto$ (λfact. λn. …), fact $\mapsto$ f, f′ $\mapsto$ (λn. …),

fact $\mapsto$ $fact^{n\ !=\ 0}$

n $\mapsto$ 5

Invoke (λfact. λn. …) on fact under the guard n != 0

Illustration of the Nuggetizer

let f = λfact. λn. let r = if (n != 0) then
let fact′ = fact fact in
let r′ = fact′ (n - 1) in
n * r′
else 1
in r
in let f′ = f f in
in let z = f′ 5 in
z

redex

Abstract Call Stack

(λn. …)

Abstract Environment

f ↦ (λfact. λn. …), fact ↦ f, f′ ↦ (λn. …),
fact ↦ $\text{fact}^{n\ !=\ 0}$, fact′ ↦ (λn. …),
n ↦ 5

Pop (λfact. λn. …), and return (λn. …) to fact′

Illustration of the Nuggetizer

```
let f = λfact. λn. let r = if (n != 0) then
                               let fact′ = fact fact in
                               let r′ = fact′ (n - 1) in
                                 n * r′
                             else  1
                         in r
in let f′ = f f in
in let z = f′ 5 in
   z
```

redex

Abstract Call Stack

(λn. …)

Abstract Environment

$f \mapsto (\lambda fact.\ \lambda n.\ \ldots),\ fact \mapsto f,\ f' \mapsto (\lambda n.\ \ldots),$
$fact \mapsto fact^{n\,!=\,0},\ fact' \mapsto (\lambda n.\ \ldots),$
$n \mapsto 5$

Illustration of the Nuggetizer

```
let f = λfact. λn. let r = if (n != 0) then
                                let fact′ = fact fact in
                                let r′ = fact′ (n - 1) in
                                   n * r′
                              else  1
                       in r
in let f′ = f f in
in let z = f′ 5 in
   z
```

redex

Abstract Call Stack

(λn. …)

Abstract Environment

f ↦ (λfact. λn. …), fact ↦ f, f′ ↦ (λn. …),
fact ↦ $\text{fact}^{n\,!=\,0}$, fact′ ↦ (λn. …),
n ↦ 5, n ↦ $(n - 1)^{n\,!=\,0}$,
r′ ↦ r

Prune (ignore) the recursive invocation of (λn. …)

Illustration of the Nuggetizer

let f = λfact. λn. let r = if (n != 0) then
let fact′ = fact fact in
let r′ = fact′ (n - 1) in
n * r′
else 1
in r
in let f′ = f f in
in let z = f′ 5 in
z

redex

Abstract Call Stack

(λn. …)

Abstract Environment

f ↦ (λfact. λn. …), fact ↦ f, f′ ↦ (λn. …),
fact ↦ $fact^{n\ != 0}$, fact′ ↦ (λn. …),
n ↦ 5, n ↦ $(n - 1)^{n\ != 0}$,
r′ ↦ r

r and, transitively, r′ ***have no*** concrete bindings, as of now

r only serves as a placeholder for the return value of the recursive call

Illustration of the Nuggetizer

let f = λfact. λn. let r = if (n != 0) then
let fact′ = fact fact in
let r′ = fact′ (n - 1) in
n * r′
else 1
in r
in let f′ = f f in
in let z = f′ 5 in
z

redex

r and, transitively, r′ ***now have*** concrete bindings

Abstract Call Stack

(λn. …)

Abstract Environment

$f \mapsto (\lambda fact.\ \lambda n.\ \ldots)$, $fact \mapsto f$, $f' \mapsto (\lambda n.\ \ldots)$,
$fact \mapsto fact^{n\ !=\ 0}$, $fact' \mapsto (\lambda n.\ \ldots)$,
$n \mapsto 5$, $n \mapsto (n - 1)^{n\ !=\ 0}$,
$r' \mapsto r$, $r \mapsto (n * r')^{n\ !=\ 0}$, $r \mapsto 1^{n\ ==\ 0}$

Merge the results of the two branches, tagged with appropriate guards

Illustration of the Nuggetizer

let f = λfact. λn. let r = if (n != 0) then
let fact′ = fact fact in
let r′ = fact′ (n - 1) in
n * r′
else 1
in r

in let f′ = f f in
in let z = f′ 5 in
z

redex

Abstract Call Stack

empty

Abstract Environment

f ↦ (λfact. λn. …), fact ↦ f, f′ ↦ (λn. …),
fact ↦ fact$^{n\ != 0}$, fact′ ↦ (λn. …),
n ↦ 5, n ↦ (n - 1)$^{n\ != 0}$,
r′ ↦ r, r ↦ (n * r′)$^{n\ != 0}$, r ↦ 1$^{n\ == 0}$, z ↦ r

Pop (λn. …), and return r to z

Illustration of the Nuggetizer

```
let f = λfact. λn. let r = if (n != 0) then
                              let fact′ = fact fact in
                              let r′ = fact′ (n - 1) in
                                n * r′
                            else  1
                        in r
in let f′ = f f in
in let z = f′ 5 in
   z
```

Abstract Call Stack

empty

Abstract Environment

$f \mapsto (\lambda fact.\ \lambda n.\ \ldots)$, $fact \mapsto f$, $f' \mapsto (\lambda n.\ \ldots)$,
$fact \mapsto fact^{n\ != 0}$, $fact' \mapsto (\lambda n.\ \ldots)$,
$n \mapsto 5$, $n \mapsto (n - 1)^{n\ != 0}$,
$r' \mapsto r$, $r \mapsto (n * r')^{n\ != 0}$, $r \mapsto 1^{n\ == 0}$, $z \mapsto r$

The abstract execution terminates

Illustration of the Nuggetizer

```
let f = λfact. λn. let r = if (n != 0) then
                               let fact′ = fact fact in
                               let r′ = fact′ (n - 1) in
                                 n * r′
                             else  1
                   in r
in let f′ = f f in
in let z = f′ 5 in
   z
```

Fixed-point of the abstract environment -- observable by *rerunning* abstract execution

Abstract Call Stack

empty

Nugget

$f \mapsto (\lambda fact.\ \lambda n.\ \ldots),\ fact \mapsto f,\ f' \mapsto (\lambda n.\ \ldots),$
$fact \mapsto fact^{n\,!=\,0},\ fact' \mapsto (\lambda n.\ \ldots),$
$n \mapsto 5,\ n \mapsto (n - 1)^{n\,!=\,0},$
$r' \mapsto r,\ r \mapsto (n * r')^{n\,!=\,0},\ r \mapsto 1^{n\,==\,0},\ z \mapsto r$

Nugget: The least fixed-point of the abstract environment

Rerunning Abstract Execution

- Can also contribute new mappings
  - Especially in presence of higher-order recursive functions which themselves return functions

Illustration of Rerunning for Convergence

```
let f = λfact. λn. let r = if (n != 0) then
                           let fact′ = fact fact in
                           let r′ = fact′ (n - 1) in
                           let r″ = r′ () in
                             λx. (n * r″)
                         else λy. 1
                      in r
in let f′ = f f in
in let z = f′ 5 in
in let z′ = z () in
   z′
```

Higher-order recursive function itself returning functions

Abstract Call Stack

empty

Abstract Environment

Illustration of Rerunning for Convergence

let f = λfact. λn. let r = if (n != 0) then
let fact′ = fact fact in
let r′ = fact′ (n - 1) in
let r″ = r′ () in
λx. (n * r″)
else λy. 1
in r
in let f′ = f f in
in let z = f′ 5 in
in let z′ = z () in
z′

redex

Abstract Call Stack

(λn. …)

Abstract Environment

f ↦ (λfact. λn. …), fact ↦ f, f′ ↦ (λn. …),
fact ↦ $\text{fact}^{n\ !=\ 0}$, fact′ ↦ (λn. …),
n ↦ 5, n ↦ $(n - 1)^{n\ !=\ 0}$,
r′ ↦ r

Prune the recursive invocation of (λn. …), as before

Illustration of Rerunning for Convergence

let f = λfact. λn. let r = if (n != 0) then
  let fact′ = fact fact in
  let r′ = fact′ (n - 1) in
  let r″ = r′ () in
    λx. (n * r″)
  else λy. 1

redex

No concrete binding for r′, the analysis simply skips over the redex ‘r′ ()’

in let z′ = z () in
  z′

Abstract Call Stack

(λn. …)

Abstract Environment

f ↦ (λfact. λn. …), fact ↦ f, f′ ↦ (λn. …),
fact ↦ $fact^{n\,!=\,0}$, fact′ ↦ (λn. …),
n ↦ 5, n ↦ $(n - 1)^{n\,!=\,0}$,
r′ ↦ r

Skip over the call-site r′ ()

Illustration of Rerunning for Convergence

let f = λfact. λn. let r = if (n != 0) then
let fact′ = fact fact in
let r′ = fact′ (n - 1) in
let r″ = r′ () in
λx. (n * r″)
else λy. 1

in let z = f′ 5 in
in let z′ = z () in
z′

r′ now has concrete bindings, but no binding for r″

Abstract Call Stack

(λn. …)

Abstract Environment

f ↦ (λfact. λn. …), fact ↦ f, f′ ↦ (λn. …),
fact ↦ $\text{fact}^{n\,!=\,0}$, fact′ ↦ (λn. …),
n ↦ 5, n ↦ $(n - 1)^{n\,!=\,0}$,
r′ ↦ r, r ↦ $(\lambda x.\ n * r'')^{n\,!=\,0}$, r ↦ $(\lambda y.\ 1)^{n\,==\,0}$

Merge the results of the two branches, tagged with appropriate guards

Illustration of Rerunning for Convergence

let f = λfact. λn. let r = if (n != 0) then
let fact′ = fact fact in
let r′ = fact′ (n - 1) in
let r″ = r′ () in
λx. (n * r″)
else λy. 1
in r
in let f′ = f f in
in let z = f′ 5 in
in let z′ = z () in
z′

Abstract Call Stack

empty

Abstract Environment

f $\mapsto$ (λfact. λn. …), fact $\mapsto$ f, f′ $\mapsto$ (λn. …),
fact $\mapsto$ $\text{fact}^{n\ !=\ 0}$, fact′ $\mapsto$ (λn. …),
n $\mapsto$ 5, n $\mapsto$ $(n - 1)^{n\ !=\ 0}$,
r′ $\mapsto$ r, r $\mapsto$ $(\lambda x.\ n * r'')^{n\ !=\ 0}$, r $\mapsto$ $(\lambda y.\ 1)^{n\ ==\ 0}$,
z $\mapsto$ r, x $\mapsto$ (), y $\mapsto$ (), z′ $\mapsto$ $(n * r'')^{n\ !=\ 0}$, z′ $\mapsto$ $1^{n\ ==\ 0}$

End of the initial run

Illustration of Rerunning for Convergence

let f = λfact. λn. let r = if (n != 0) then
    let fact′ = fact fact in
    let r′ = fact′ (n - 1) in
    let r″ = r′ () in
      λx. (n * r″)
  else λy. 1
in r
in let f′ = f f in
in let z = f′ 5 in
in let z′ = z () in
  z′

redex

r′ has concrete bindings

Abstract Call Stack

(λn. …)

Abstract Environment

f ↦ (λfact. λn. …), fact ↦ f, f′ ↦ (λn. …),
fact ↦ $\text{fact}^{n\ !=\ 0}$, fact′ ↦ (λn. …),
n ↦ 5, n ↦ $(n - 1)^{n\ !=\ 0}$,
r′ ↦ r, r ↦ $(\lambda x.\ n * r'')^{n\ !=\ 0}$, r ↦ $(\lambda y.\ 1)^{n\ ==\ 0}$,
z ↦ r, x ↦ (), y ↦ (), z′ ↦ $(n * r'')^{n\ !=\ 0}$, z′ ↦ $1^{n\ ==\ 0}$,
x ↦ $()^{n\ !=\ 0}$, y ↦ $()^{n\ !=\ 0}$, r″ ↦ $(n * r'')^{n\ !=\ 0}$, r″ ↦ $1^{n\ ==\ 0}$

During the rerun

Illustration of Rerunning for Convergence

let f = λfact. λn. let r = if (n != 0) then
    let fact′ = fact fact in
    let r′ = fact′ (n - 1) in
    let r″ = r′ () in
      λx. (n * r″)
  else λy. 1
in r

in let f′ = f f in
in let z = f′ 5 in
in let z′ = z () in
  z′

Now a fixed-point of the abstract environment -- observable by *rerunning* abstract execution

Abstract Call Stack

empty

Nugget

f ↦ (λfact. λn. …), fact ↦ f, f′ ↦ (λn. …),
fact ↦ $\text{fact}^{n\,!=\,0}$, fact′ ↦ (λn. …),
n ↦ 5, n ↦ $(n - 1)^{n\,!=\,0}$,
r′ ↦ r, r ↦ $(\lambda x.\ n * r'')^{n\,!=\,0}$, r ↦ $(\lambda y.\ 1)^{n\,==\,0}$,
z ↦ r, x ↦ (), y ↦ (), z′ ↦ $(n * r'')^{n\,!=\,0}$, z′ ↦ $1^{n\,==\,0}$,
x ↦ $()^{n\,!=\,0}$, y ↦ $()^{n\,!=\,0}$, r″ ↦ $(n * r'')^{n\,!=\,0}$, r″ ↦ $1^{n\,==\,0}$

End of the rerun

However…

Number of reruns required to reach a fixed-point is always (*provably*) finite

- Abstract environment is monotonically increasing across runs
- Size of abstract environment is strongly bound
  - Domain, range and guards of all mappings are fragments of the source program

All feasible mappings will eventually be collected after some finite number of reruns, and a fixed-point reached

Properties of the Nuggetizer

Soundness Nugget denotes all values that may arise in variables at runtime

Termination Nuggetizer computes a nugget for all programs

Runtime Complexity Runtime complexity of the nuggetizer is $O(n! \cdot n^3)$, where n is the size of a program

- We expect it to be significantly less in practice

Related Work

- No direct precedent to our work
  - *An automated algorithm for abstracting arbitrary higher-order programs as first-order inductive definitions*
- A logical descendent of 0CFA [Shivers'91]
- Dependent, Refinement Types [Xi+'05, Flanagan+'06]
  - Require programmer annotations
    - Our approach: No programmer annotations
- Logic Flow Analysis [Might'07]
  - Does not generate inductive definitions
  - Invokes theorem prover many times, and on-the-fly
    - Our approach: only once, at the end

Currently working towards

- Completeness
  - *A lossless* translation of higher-order programs to first-order inductive definitions

(The current analysis is sound but not complete)

- Incorporating Flow-Sensitive Mutable State
  - Shape-analysis of heap data structures
- Prototype Implementation

Thank You

Example of Incompleteness

Inspired by bidirectional bubble sort

```
let f = λsort. λx. λlimit. if (x < limit) then
                              sort sort (x + 1) (limit - 1)
                           else  1
in f f 0 9
```

Range of x is [0, 5] and range of limit is [4, 9]

Nugget at x and limit:

$$\{ x \mapsto 0, x \mapsto (x + 1)^{x < limit}, limit \mapsto 9, limit \mapsto (limit - 1)^{x < limit} \}$$

$$\Downarrow$$

$$\{ x \mapsto 0, \ldots, x \mapsto 9, limit \mapsto 9, \ldots, limit \mapsto 0 \}$$

Correlation between order of assignments to x and limit is lost

External Inputs

let f = λfact. λn. if (n != 0) then
    n * fact fact (n - 1)
  else 1
in if (***inp*** $\geq$ 0) then
  f f ***inp***

Property of interest: Symbolic range of n is [0, …, ***inp***]

Nugget at n: $\{ n \mapsto \boldsymbol{inp}^{\boldsymbol{inp} \geq 0}, n \mapsto (n - 1)^{n \,!= 0} \}$

$$\Downarrow$$

$$\{ n \mapsto \boldsymbol{inp}, n \mapsto \boldsymbol{inp} - 1, \ldots, n \mapsto 0 \}$$

A more complex example

$Z = \lambda f.\ (\lambda x.\ f\ (\lambda y.\ x\ x\ y))\ (\lambda x.\ f\ (\lambda y.\ x\ x\ y))$

```
let f′ = λfact. λn. if (n != 0) then
                          n * fact (n - 1)
                        else  1
in Z f′ 5
```

Nugget at n:

$\{\ n \mapsto 5,\ n \mapsto y,\ y \mapsto (n - 1)^{n\ != 0}\ \} \equiv \{\ n \mapsto 5,\ n \mapsto (n - 1)^{n\ != 0}\ \}$

Another complex example

```
let g = λfact′. λm. fact′ fact′ (m - 1) in
let f = λfact. λn. if (n != 0) then
                        n * g fact n
                      else  1
in f f 5
```

Nugget at n and m: $\{ n \mapsto 5,\ m \mapsto n^{n\,!=\,0},\ n \mapsto (m - 1) \}$

$$\Downarrow$$

$$\{ n \mapsto 5,\ n \mapsto 4,\ n \mapsto 3,\ n \mapsto 2,\ n \mapsto 1,\ n \mapsto 0 \}$$

$$\{ m \mapsto 5,\ m \mapsto 4,\ m \mapsto 3,\ m \mapsto 2,\ m \mapsto 1 \}$$

General, End-to-End Programming Logic

```
let f = λfact. λn. assert (n ≥ 0);
                   if (n != 0) then
                     n * fact fact (n - 1)
                   else  1
in f f 5
```

assert (n ≥ 0) would be compiled down to a theorem, and automatically proved by the theorem prover over the automatically generated nugget

Many asserts are implicit

- Array bounds and null pointer checks

Methodology by Analogy

	Program Model Checking	Our Approach
Abstraction Model	Finite Automaton	First-Order Inductive Definitions (Nugget)
Verification Method	Model Checking	Theorem Proving
Pros	Faster	Higher-Order Programs, Inductive Properties
Cons	First-Order Programs, Non-Inductive Properties	Slower